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Abstract. In a wide-open quantum system, the effect of the Hamiltonian is negligible by
comparison with the effect of the environment. For open systems, this is the opposite limit
to closed or isolated systems. The quantum state diffusion model provides equations for the
localization or reduction of quantum states of wide-open systems. The ensemble localization of
an operator is defined, and it is proved that the rate of selflocalization of a selfadjoint environment
operator towards one of its eigenstates is no less than 2. A bound is also obtained for the rate
of selflocalization of some non-selfadjoint operators, which localize to minimum indeterminacy
wave packets. The theory is presented for quasiclassical systems. For a sufficient number
of independent environment operators, the states localize asymptotically to wave packets with
Heisenberg indeterminacy products close to the minimum, which look to classical eyes like phase
space points. To zeroth order in $\hbar$, the time-dependent WKB theory of quantum state diffusion
due to a single operator shows localization or reduction within and between fixed classical sheets
or Lagrangian manifolds. To first order, the sheets themselves diffuse. The rate of localization
in an ensemble is determined by commutation terms with either sign and by correlation terms
which always increase the localization. For the quasiclassical case the latter dominate, and this
leads to a purely classical theory of localization, with a picture based on the diffusion of phase
space densities. This means that state diffusion dynamics, like Hamiltonian dyanamics, has a
purely classical form, in which Planck's constant plays no role.

1. Introduction

The quantum state diffusion model ([1–3], see also [4–15]) is a formulation of the quantum
mechanics of open systems which abandons locality but keeps other aspects of classical
physics intact, as discussed in the second reference of [25]. This and the related quantum
jump or stochastic wavefunction methods [13, 16–22] have established that:

(i) Ordinary quantum mechanics is no longer the best for *all* practical purposes [25],
transforming the situation lamented by Bell [23] a few years ago. Although the new models
depend on ordinary quantum mechanics for their formulation, they provide very different
pictures of what happens to an *open* quantum system.

(ii) In these models, the changing state of a single quantum system is represented directly
by a stochastically evolving pure state vector, as for a single run of a laboratory experiment.
The state vector is then not merely a device for computing statistical expectations in an
ensemble of systems. Such direct representations have a long history, starting with de
Broglie pilot wave theory, but the new models have established their practical and intuitive
value.

(iii) In the state diffusion model, quantum measurement is nothing special. It is merely
an example of the interaction of a system with its environment. This interaction of a

system with measuring apparatus can be treated in the same way, and using the same basic equations, as the interaction with any other environment, such as a heat bath. These other environments also produce localization or reduction, in the absence of anything resembling a measurement. This is consistent with the ideas of Joos and Zeh [24, 25], but the model differs in having an explicit equation for the localization, modifying the Schrödinger equation.

(iv) Localization or reduction appears on distinguishing between quantum expectations for an individual quantum system and ensemble means over properties of members of an ensemble of quantum systems. These properties are functions of quantum expectations. In the theory of localization or reduction, the functions can be linear, quadratic, logarithmic, or of any other form. Quadratic functions are particularly important.

(v) For this purpose, density operators are inadequate. They can represent ensemble means over linear functions of the quantum expectations only.

In the quantum state diffusion model of open quantum systems, an individual system is represented by a pure state vector which diffuses in the state space consistently with the Bloch or master equation for the density operator of an ensemble [1–3]. The state vector satisfies the unique equation (2.14). The model provides a more concrete representation of the quantum dynamics of individual systems than ordinary quantum mechanics, and sometimes leads to more efficient computations, but does not lead to different results.

Although the state diffusion model provides both insight and practical tools for the solution of physical problems, it cannot be regarded as a complete and satisfactory alternative fundamental theory of quantum mechanics, as it depends on an arbitrary division or cut between system and environment, and in practice on approximations that have to be made as a result of this division.

The detailed analysis here is restricted to *wide-open* systems, that is, to open systems which interact with the environment so strongly that the effect of the system Hamiltonian H is negligible by comparison with the effect of the environment, so H can be set equal to zero. This is not so restrictive as it might seem.

It can be applied, for example, to the interaction between a system and measuring apparatus in all those situations in which a measurement is well represented by von Neumann's projection postulate [26], for it is then assumed that the dynamical variables being measured are not changed significantly by the internal dynamics of the system whilst the measurement is taking place, as in a well designed experiment. It can be applied more generally using interaction representation.

In condensed matter physics, chemical physics and biophysics, many quantum systems interact very strongly with their fluid or solid environments. The theoretical treatment of such systems has been impeded by the inadequacy of ordinary quantum theory of open systems based on the density operator, a situation that could be remedied by the practical use of state diffusion methods, whose main applications so far have been to quantum optics.

Localization as defined here is a property of the state vectors or wave packets of open systems in state diffusion theory [2], not to be confused with the use of the term in other contexts, such as Anderson localization. As shown here, the states of most wide-open systems become strongly localized in phase space, with state vectors represented by wave packets with near to the minimum Heisenberg indeterminacy, so their theory is a zeroth-order approximation to an important class of problems in which these wave packets closely but not exactly follow the Hamiltonian trajectories of classical mechanics. This is a different quasiclassical limit of quantum mechanics than the WKB limit, which applies to closed or isolated quantum systems.

The results presented here generalize and extend some of the work on the interaction of free particles by Diósi [26] and by Gutarek and Gisin [8], on dissipative systems by Salama

and Gisin [10], on reduction by Gisin [7] and on measurement [1, 2]. A quasiclassical theory of some related state diffusion equations appears in [29].

Section 2 introduces the basic definitions and state vector diffusion equations. Section 3 derives the localization of a dynamical variable in terms of commutators and quantum correlations, showing that the rate of ensemble selflocalization of a selfadjoint environment operator is no less than 2. Section 4 extends the localization theory to conjugate environment operators and their asymptotic minimum indeterminacy wave packets.

The dynamically linear theory of localization by operators with constant commutators appears in section 5. A relative simplicity of this theory allows a bound on the quantum correlations for the asymptotic states when there are an arbitrary finite number of environment operators L_j. Section 6 introduces the theory of localization by non-selfadjoint operators, which can be used to represent dissipative systems, and obtains a bound on their rate of selflocalization to wave packets.

The quasiclassical theory follows in section 7, where the state vector is approximated by a phase space distribution $D(x, y)$. The more detailed theory for the WKB representation is given in section 8.

Section 9 puts the theory in context, with particular emphasis on the existence of a purely classical theory of localization or reduction. Bounds on quantum mean square deviations are given in appendix 1 and equivalent sets of environment operators are treated in appendix 2.

2. Definitions and state diffusion equations

The current pure state of the system is $|\psi\rangle$. G is an operator, which may or may not be selfadjoint. Capital letters are used for operators and small letters for complex numbers. The quantum expectation of G for the state $|\psi\rangle$ is

$$g = \langle G \rangle = \langle \psi | G | \psi \rangle. \tag{2.1}$$

This notation will be used even when G is not selfadjoint, as in the case of creation and annihilation operators. For selfadjoint G, the quantum expectation of the operator is the expectation of the dynamical variable.

The corresponding shifted operator, with zero expectation, is

$$G_\Delta = G - \langle G \rangle. \tag{2.2}$$

Shifted and unshifted operators give the same value in commutators

$$[G, B] = [G_\Delta, B]. \tag{2.3}$$

This relation will be used quite frequently.

A general operator G may be separated into its real and imaginary (strictly selfadjoint and skewadjoint) parts G_R and iG_I. For the usual annihilation operator A of an oscillator, these parts are given by

$$X/(2\hbar)^{1/2} \qquad iY/(2\hbar)^{1/2} \tag{2.4}$$

where X and Y are convenient conjugate dynamical variables, and A, $A^\dagger$ and X, Y satisfy the standard commutation relations.

The effect of a set of environment operators is unchanged under unitary transformation in the operator space [1], so that it helps to use definitions and equations that respect this invariance. One such definition is the skew commutator

$$[B^\dagger, C] = i([B_R, C_I] + [C_R, B_I]) \tag{2.5}$$

so that the skew commutator of B with itself is non-zero, unless it is selfadjoint to within a phase factor:

$$[B^\dagger, B] = 2i[B_R, B_I]. \tag{2.6}$$

The quantum correlation of two operators for the state $|\psi\rangle$ is extensively used in the following sections. It is defined as

$$\sigma(B, C) = \langle B_\Delta^\dagger C_\Delta \rangle = \langle B_\Delta^\dagger C \rangle = \langle B^\dagger C_\Delta \rangle = \langle B^\dagger C \rangle - \langle B \rangle^* \langle C \rangle \tag{2.7}$$

from which it follows that

$$\sigma(B, C) = \sigma(B_R, C_R) + \sigma(B_I, C_I) + i\langle B_{\Delta R} C_{\Delta I} - C_{\Delta R} B_{\Delta I}\rangle$$
$$\tfrac{1}{2}(\sigma(B, C) + \sigma(C, B)) = \sigma(B_R, C_R) + \sigma(B_I, C_I) + \tfrac{1}{2}i\langle [B_R, C_I] + [C_R, B_I]\rangle$$
$$\sigma(B, B) = \sigma(B_R, B_R) + \sigma(B_I, B_I) + i\langle [B_R, B_I]\rangle \tag{2.8}$$
$$\qquad = \sigma(B_R, B_R) + \sigma(B_I, B_I) + \tfrac{1}{2}\langle [B^\dagger, B]\rangle$$
$$\tfrac{1}{2}(\sigma(B, B) + \sigma(B^\dagger, B^\dagger)) = \sigma(B_R, B_R) + \sigma(B_I, B_I).$$

The quantum correlation of B with itself is just the quantum mean square (QMS) deviation of B:

$$\sigma(B, B) = \langle B^\dagger B \rangle - \langle B \rangle^* \langle B \rangle = \sigma^2(B). \tag{2.9}$$

Note that the QMS deviation $\sigma^2(B)$ of a non-selfadjoint operator B is not the same as that of its adjoint $B^\dagger$, but that their mean is the sum of the QMS deviations $\sigma^2(B_R)$, $\sigma^2(B_I)$ of the real and imaginary parts. This convention is helpful to the later analysis, but differs from an earlier convention of Caves [27].

In particular, for annihilation and creation operators

$$\sigma^2(A) + \tfrac{1}{2} = \sigma^2(A^\dagger) - \tfrac{1}{2} = \tfrac{1}{2}\hbar\left(\sigma^2(X) + \sigma^2(Y)\right) \tag{2.10}$$

and for selfadjoint B, $\sigma^2(B) = (\Delta b)^2$.

The ensemble mean of a property $\pi(\psi)$ of the states $|\psi\rangle$ of an ensemble is denoted by $M(\pi)$. The ensemble localization Λ of a selfadjoint operator B is defined here as the inverse of the ensemble mean of the QMS deviation:

$$\Lambda = \left(M\sigma^2(B)\right)^{-1} = (M\sigma(B, B))^{-1} = \left(M(\langle B^\dagger B \rangle - \langle B \rangle^* \langle B \rangle)\right)^{-1} \qquad (B \text{ sadj}). \tag{2.11}$$

Its importance is that with this definition there are strict bounds on the rate of localization, as shown in section 3. For non-selfadjoint operators B it is more convenient to use a definition which is symmetric in B and $B^\dagger$, which is given by

$$\Lambda^{-1} = M\left(\sigma^2(B_R) + \sigma^2(B_I)\right) = M\left(\sigma^2(B) - \tfrac{1}{2}\langle [B^\dagger, B]\rangle\right). \tag{2.12}$$

These definitions are used to obtain bounds on rates of localization in sections 3 and 6.

The state diffusion theory replaces the deterministic evolution of the density operator ρ representing an ensemble of wide-open systems,

$$\dot{\rho} = \sum_j \left(L_j \rho L_j^\dagger - \tfrac{1}{2} L_j^\dagger L_j \rho - \tfrac{1}{2} \rho L_j^\dagger L_j \right) \tag{2.13}$$

by a unique stochastic diffusion of a quantum state, representing an individual system of the ensemble in interaction with its environment [2]. Because the systems are wide open, there is no Hamiltonian term. The corresponding quantum state diffusion equation is a stochastic differential equation for the normalized state vector $|\psi\rangle$, whose differerential Itô form is, from [2],

$$|\,d\psi\rangle = -\tfrac{1}{2} \sum_j (L_j^\dagger L_j + \ell_j^* \ell_j - 2\ell_j^* L_j)|\psi\rangle \, dt + \sum_j (L_j - \ell_j)|\psi\rangle \, d\xi_j \tag{2.14}$$

where the ℓ_j are defined by (2.1).

From the density operator theory, or directly from appendix 2, L_j may be chosen to be linearly independent. In the Itô theory, the stochastic fluctuation or noise of the diffusion process is all contained in the standard normalized Wiener fluctuation terms $d\xi_j$, which are of order $(dt)^{1/2}$ and which satisfy the relations

$$d\xi_j \, d\xi_k = 0 \qquad d\xi_j^* \, d\xi_k = \delta_{jk} dt \tag{2.15}$$

$$M \, d\xi_j = 0. \tag{2.16}$$

Note that the quadratic relations (2.15) apply to individual states before taking an ensemble mean. This is because in the Itô formalism for stochastic differentials, by analogy with the theory of ordinary differentials, corrections for finite dt that become negligible in the limit as dt tends to zero are omitted from the equations. Thus any terms of order $(dt)^{3/2}$ are negligible, including all products of the form $dt \, d\xi$, and fluctuating terms with zero mean are negligible if they are of order dt. For example, in equation (2.15), $|d\xi_j|^2 = dt$ says that the fluctuation

$$|\,d\xi_j|^2 - M|\,d\xi_j|^2 = |\,d\xi_j|^2 - dt$$

with zero mean is of order dt, and does not imply an exact equality for finite changes. The quadratic relations (2.15) are the complex form of the equations (4.3.15a) and (4.3.15c) of Gardiner [28], where more details on the Itô formalism may be found.

Unlike (2.15), the simple linear relation (2.16) is valid only for the ensemble mean.

Despite the appearance of quadratic differentials in the Itô theory, the additivity rule, by which contributions from independent differential drift and diffusion terms may be added together to obtain their combined effect, is maintained over a time dt. This is because of the statistical independence of the stochastic fluctuations expressed by the δ-function in equation (2.15). So in these differential stochastic diffusion equations, the detailed derivations are carried out for single environment operator L. When there are many L_j, their effects for time dt can be added by using the additivity rule.

Most of the equations and results refer to an initial ensemble of identical pure states, which diffuse in time dt to an ensemble of different states. For this reason means over the ensemble for changes in time dt on the left side of equations often contain an ensemble mean M, whereas the right side, which depends only on the initial state, does not. For equations representing the more general situation in which the initial ensemble is of differing states, it is only necessary to take an ensemble mean of the right side also.

3. Localization of a dynamical variable

In this section the general theory of diffusion and localization is introduced through a simple example of diffusion and localization of a selfadjoint operator G representing a dynamical variable by an environment whose effect is represented by a single selfadjoint operator L. In this case $\sigma^2(G) = (\Delta g)^2$.

Then the change in $\langle G \rangle$ due to the diffusion of $|\psi\rangle$ in time dt is

$$d\langle G \rangle = \langle \psi|G|d\psi \rangle + \langle d\psi|G|\psi \rangle + \langle d\psi|G|d\psi \rangle$$
$$= -\tfrac{1}{2}\langle GL_\Delta^2 + L_\Delta^2 G - 2L_\Delta GL_\Delta \rangle dt + \langle GL_\Delta \rangle\, d\xi + \langle L_\Delta G \rangle\, d\xi^* \quad (3.1)$$
$$= -\tfrac{1}{2}\langle [[G, L], L] \rangle dt + \sigma(G, L)\, d\xi + \sigma(L, G)\, d\xi^*$$

which is a sum of three terms: a drift given by the commutator and two diffusion terms whose magnitude is proportional to the quantum correlation of G and L for the state $|\psi\rangle$.

The mean rate of change of the QMS deviation is therefore

$$M(d\sigma^2(G)/dt) = M\left(d\langle G^2 \rangle - 2\langle G \rangle\, d\langle G \rangle - (d\langle G \rangle)^2\right)/dt$$
$$= -\tfrac{1}{2}\langle [[G^2, L_\Delta], L_\Delta] \rangle + \langle G \rangle\langle [[G, L_\Delta], L_\Delta] \rangle - 2|\langle GL_\Delta \rangle|^2$$
$$= -\tfrac{1}{2}\langle [[G^2, L], L] \rangle + \langle G \rangle\langle [[G, L], L] \rangle - 2|\sigma(G, L)|^2$$
$$= -\tfrac{1}{2}\langle [[G_\Delta^2, L], L] \rangle - 2|\sigma(G, L)|^2$$
$$= -\operatorname{Re}\langle G_\Delta[G, L]L + [G, L]G_\Delta L \rangle - 2|\sigma(G, L)|^2 \quad (3.2)$$

which is the sum of a commutation term, which can have either sign, and a non-positive term which is proportional to the square of the quantum correlation of G and L.

If the commutator

$$[G, L] = i\gamma \quad (3.3)$$

with real constant γ, then

$$M(d\sigma^2(G)/dt) = 2\operatorname{Im}(\gamma\sigma(G, L)) - 2|\sigma(G, L)|^2. \quad (3.4)$$

If G commutes with L, then the commutators are zero,

$$M(d\sigma^2(G)/dt) = -2|\sigma(G, L)|^2 \quad (3.5)$$

and the ensemble mean of the QMS deviation of G decreases, except when the correlation is zero. So a selfadjoint environment operator tends to localize any commuting dynamical operator.

In particular, if $G = L$, the rate of selfdiffusion of $\langle L \rangle$ is

$$d\langle L \rangle = 2\operatorname{Re}(\langle LL_\Delta \rangle\, d\xi) = 2\sigma^2(L)\, d\xi_R, \quad (3.6)$$

so the diffusion rate is proportional to the QMS deviation and localized operators diffuse more slowly. The mean change in the QMS deviation is

$$M(d\sigma^2(L)/dt) = -2|\langle L^2 \rangle - \langle L \rangle^2|^2 = -2(\sigma^2(L))^2. \quad (3.7)$$

Essentially this last result has been obtained by Gisin [7]. To go further, it is necessary to consider the properties of the evolving ensemble, rather than those of the individual state, using the ensemble localization defined in section 2. Take an ensemble mean over both sides of (3.7), and use the Schwarz inequality for means to obtain

$$\frac{d}{dt} M\sigma^2(L) = -2M(\sigma^2(L))^2 \leqslant -2(M\sigma^2(L))^2. \tag{3.8}$$

So a bound on the change of the ensemble localization of L defined in section 2 as

$$\Lambda = \Lambda(L, t) = (M\sigma^2(L))^{-1} \tag{3.9}$$

is given by

$$\frac{d}{dt}(\Lambda^{-1}) \leqslant -2\Lambda^{-2} \tag{3.10}$$

$$\frac{d\Lambda}{dt} \geqslant 2. \tag{3.11}$$

That is, the rate of change of the ensemble localization is greater than 2. The right side is dimensionless because in equation (2.14) all environment operators have dimension $[\text{time}]^{-1/2}$, and so their QMS deviation has dimension $[\text{time}]^{-1}$.

If there are many environment operators, all of which commute, then by (3.5), each tends to localize the others, so the inequality (3.11) still holds. However, the localization rate can be reduced by non-commuting environment operators, as shown in the next section.

4. Conjugate operators

This section treats the effect of conjugate selfadjoint operators on drift, diffusion and localization. Let X_0 and Y_0 be the operators, with

$$[X_0, Y_0] = i\hbar. \tag{4.1}$$

Suppose there are two environment operators

$$L_1 = \alpha_1 X_0 \qquad L_2 = \alpha_2 Y_0 \tag{4.2}$$

where it can be assumed without losing generality that both α_j are positive. It is then more convenient to use the scaled conjugate operators X, Y, such that

$$X_0 = (\alpha_2/\alpha_1)^{1/2}X \qquad Y_0 = (\alpha_1/\alpha_2)^{1/2}Y \tag{4.3}$$

and the constant $\alpha = (\alpha_1\alpha_2)^{1/2}$, so that

$$L_1 = \alpha X \qquad L_2 = \alpha Y. \tag{4.4}$$

For the drift of $\langle X \rangle$ and $\langle Y \rangle$, the double commutator in equation (3.1) is zero, so that

$$d\langle X \rangle = \alpha^2 \left(2\sigma^2(X)\, d\xi_{1R} + 2\,\mathrm{Re}(\sigma(X, Y)\, d\xi_2)\right)$$
$$d\langle Y \rangle = \alpha^2 \left(2\sigma^2(Y)\, d\xi_{2R} + 2\,\mathrm{Re}(\sigma(Y, X)\, d\xi_1)\right). \tag{4.5}$$

In the following, assume for simplicity that $\langle X \rangle = \langle Y \rangle = 0$. For the QMS deviations of X and Y, the commutation relations give

$$[[X^2, Y], Y] = [[Y^2, X]] = -2\hbar^2 \qquad (4.6)$$

so from (3.2) and the additivity rule, the changes in the QMS deviations are given by

$$M\,d\sigma^2(X)/dt = \alpha^2(\hbar^2 - 2(\sigma^2(X))^2 - 2|\sigma(X, Y)|^2)$$

$$M\,d\sigma^2(Y)/dt = \alpha^2(\hbar^2 - 2(\sigma^2(Y))^2 - 2|\sigma(Y, X)|^2) \qquad (4.7)$$

$$M\frac{d}{dt}\left(\sigma^2(X) + \sigma^2(Y)\right) = 2\alpha^2\left(\hbar^2 - (\sigma^2(X))^2 - (\sigma^2(Y))^2 - 2|\sigma(X, Y)|^2\right).$$

From the inequalities (A1.6) and (A1.8) of appendix 1, the QMS deviation and correlation terms are together at least as great as $2\alpha^2\hbar^2$, so the right side is never positive. The ensemble localization continues with asymptotic approach to a minimum indeterminacy Gaussian wave packet with equal QMS deviations for X and for Y:

$$\sigma^2(X) = \sigma^2(Y) = \langle X^2 \rangle = \langle Y^2 \rangle = \hbar/2. \qquad (4.8)$$

In this special representation, the wave packet is symmetric under rotations in the phase space, but for general conjugate variables, obtained from X and Y by an arbitrary linear canonical phase space transformation, the asymptotic wave packet could be any kind of squeezed state.

As the asymptotic wave packet is approached, the quantum expectations of X and Y continue to diffuse in phase space, according to equations (4.5).

5. Dynamically linear localization theory

A system has m freedoms, and $2m$ operators X_r, Y_s, which could be positions and momenta, or quadratures of amplitudes of an electromagnetic field, with the standard commutators

$$[X_r, Y_s] = i\hbar\delta_{rs} \qquad [X_r, X_s] = [Y_r, Y_s] = 0. \qquad (5.1)$$

A dynamically linear operator G has the form

$$G = \sum_r (b_r X_r + c_r Y_r) \qquad (5.2)$$

where b_r, c_r are arbitrary complex constants. The essential property of dynamically linear operators G_j, required for this section, is that any pair of them should satisfy

$$[G_j, G_k] = \text{constant}. \qquad (5.3)$$

Let an operator G and environment operators L_j be selfadjoint dynamically linear operators, representing dynamical variables, so that equation (3.4) can be used, since by the additivity rule of section 2, it generalizes to any number of environment operators L_j. So if

$$[G, L_j] = i\gamma_j \qquad (5.4)$$

$$\sigma_j = \sigma(L_j, G)^* = \sigma(G, L_j) \qquad (5.5)$$

the mean change in the QMS deviation is

$$\text{M} \, d\sigma^2(G)/dt = 2 \operatorname{Im} \sum_j \sigma_j^* \gamma_j - 2 \sum_j |\sigma_j|^2. \tag{5.6}$$

Note that σ depends on the state, but that γ does not.

The vectors σ and γ are defined as

$$\sigma = (\sigma_1, \sigma_2, \ldots) \qquad \gamma = (\gamma_1, \gamma_2, \ldots) \tag{5.7}$$

so that with the usual definition of the complex scalar product,

$$\text{M} \, d\sigma^2(G)/dt = 2 \operatorname{Im}(\sigma, \gamma) - 2(\sigma, \sigma). \tag{5.8}$$

Asymptotic states which no longer localize must therefore have the right side of (5.8) zero and so satisfy the inequalites

$$|\sigma|^2 \leqslant |\gamma||\sigma|, \qquad |\sigma| \leqslant |\gamma|. \tag{5.9}$$

From the definitions (5.4, 5.5),

$$\sum_j |\sigma(G, L_j)|^2 \leqslant \sum_j |[G, L_j]|^2 \qquad \text{(asymptotic)}. \tag{5.10}$$

The left-hand side depends on the state, and the right-hand side is a constant of order $\hbar^2$, because the operators are linear in the conjugate variables. The equation has been written in a form that is independent of the original representation. It is a general result that applies to the asymptotic states of localization of an *arbitrary* selfadjoint dynamically linear operator G. If the L_j span the phase space of a system of m freedoms, then these inequalities confine asymptotic states to regions of approximate volume $\hbar^m$.

6. Dissipation, non-selfadjoint operators

For most interactions with the environment, including all those with dissipation, the environment operators L_j are not all selfadjoint.

Here are some general results on the change with time of the ensemble mean M of the quantum mean square deviations given by

$$\sigma^2(G) = \langle G^\dagger G \rangle - |\langle G \rangle|^2 \tag{6.1}$$

of an arbitrary operator G, not necessarily selfadjoint, by environment operators L that also need not be selfadjoint. Because of the additivity rule given in section 2, the results for an arbitrary number of environment operators follow from the result for one environment operator, for which the Itô equation is

$$|d\psi\rangle = -\tfrac{1}{2}(L^\dagger L + \ell^* \ell - 2\ell^* L)|\psi\rangle dt + (L - \ell)|\psi\rangle \, d\xi. \tag{6.2}$$

Using the shifted L_Δ operator, the equation takes a slightly simpler form:

$$|d\psi\rangle = -\tfrac{1}{2}\langle L_\Delta^\dagger L_\Delta + \ell L_\Delta^\dagger - \ell^* L_\Delta\rangle|\psi\rangle dt + L_\Delta|\psi\rangle \, d\xi. \tag{6.3}$$

The change in the expectation of G is then given by

$$
\begin{aligned}
d\langle G \rangle &= -\tfrac{1}{2}\langle L_\Delta^\dagger L_\Delta G + G L_\Delta^\dagger L_\Delta - 2 L_\Delta^\dagger G L_\Delta \\
&\quad + \ell G L_\Delta^\dagger + \ell^* L_\Delta G - \ell^* G L_\Delta - \ell L_\Delta G\rangle dt + \langle G L_\Delta \rangle\, d\xi + \langle L_\Delta^\dagger G \rangle\, d\xi^* \\
&= -\tfrac{1}{2}\langle L^\dagger[L, G] + [G, L^\dagger] L\rangle dt + \sigma(G^\dagger, L)\, d\xi + \sigma(L, G)\, d\xi^*
\end{aligned}
\tag{6.4}
$$

where commutators with L and L_Δ are the same. If G is selfadjoint this simplifies to

$$
d\langle G \rangle = \mathrm{Re}\langle L^\dagger[G, L]\rangle dt + 2\,\mathrm{Re}\,(\sigma(G, L)\, d\xi) \qquad (G \ \text{sadj}).
\tag{6.5}
$$

Notice that in general the diffusion terms produce a drift in the mean value in addition to the complex diffusion. The drift is zero if G commutes with L and $L^\dagger$.

The ensemble mean of the change in the QMS deviation of G is

$$
\mathrm{M}\, d\sigma^2(G) = \mathrm{M}\left(d\langle G^\dagger G\rangle - 2\,\mathrm{Re}(\langle G\rangle^*\, d\langle G\rangle) - d\langle G\rangle^*\, d\langle G\rangle\right)
$$

so using (6.4) and the fact that $G^\dagger G$ is selfadjoint we obtain

$$
\begin{aligned}
\mathrm{M}\frac{d\sigma^2(G)}{dt} &= -\mathrm{Re}\langle L^\dagger[L, G^\dagger G] - g^*\langle L^\dagger[L, G] + [G, L^\dagger]L\rangle \\
&\quad - \left(|\sigma(G^\dagger, L)|^2 + |\sigma(L, G)|^2\right) \\
&= -\mathrm{Re}\langle L^\dagger[L, G^\dagger]G + L^\dagger G^\dagger[L, G] - g L^\dagger[L, G] - g^* L^\dagger[L, G]\rangle \\
&\quad - \left(|\sigma(G^\dagger, L)|^2 + |\sigma(L, G)|^2\right) \\
&= \mathrm{Re}\langle L^\dagger[G^\dagger, L]G_\Delta + L^\dagger G_\Delta^\dagger[G, L]\rangle - \left(|\sigma(G^\dagger, L)|^2 + |\sigma(L, G)|^2\right).
\end{aligned}
\tag{6.6}
$$

This result generalizes to an arbitrary number of environment operators L_j, by summing over terms with L_j.

If L commutes with G and $G^\dagger$, then all the commutators go to zero, and the right side of the equation is non-positive, so that the QMS deviation of G is non-increasing and normally decreases. It follows directly from additivity that the same applies when there are any number of environment operators L_j, provided G commutes with all $L_j, L_j^\dagger$.

When the commutators

$$
[G, L] = -2\gamma \qquad [G^\dagger, L] = -2\gamma_0
\tag{6.7}
$$

are constants as defined, as in the case when both L and G are linear combinations of coordinates and conjugate momenta, the diffusion of $\langle G\rangle$ and the mean change in the QMS deviation are particularly simple. They are

$$
d\langle G\rangle = -(\ell\gamma_0^* + \ell^*\gamma) + \sigma(G^\dagger, L)\, d\xi + \sigma(L, G)\, d\xi^*
\tag{6.8}
$$

$$
\mathrm{M}\, d\sigma^2(G)/dt = -2\,\mathrm{Re}\left(\gamma_0\sigma(L, G) + \gamma\sigma(L, G^\dagger)\right) - \left(|\sigma(G^\dagger, L)|^2 + |\sigma(L, G)|^2\right).
\tag{6.9}
$$

The next result is on the rate of selflocalization of a non-selfadjoint operator L when the commutator $[L^\dagger, L]$ is constant. From equation (6.9), if $G = L$, we have

$$
\gamma = 0 \qquad [L^\dagger, L] = -2\gamma_0 \ \ (\text{real}) \qquad \sigma(L, L) = \sigma^2(L) \ \ (\text{real})
\tag{6.10}
$$

and

$$\frac{d}{dt}M\sigma^2(L) = -2\gamma_0\sigma^2(L) - (\sigma^2(L))^2 - |\sigma(L^\dagger, L)|^2. \tag{6.11}$$

For an ensemble take the mean of the right-hand side and use Schwarz's inequality to obtain

$$\frac{d}{dt}M\sigma^2(L) \leqslant -2\gamma_0 M\sigma^2(L) - (M\sigma^2(L))^2 \tag{6.12}$$

where the negative term $-|\sigma(L^\dagger, L)|^2$ has been omitted. This omission makes the inequality weaker than inequality (3.11) for selfadjoint operators when L is close to being selfadjoint.

With the definition (2.12) for the ensemble localization Λ,

$$\frac{d\Lambda^{-1}}{dt} \leqslant \gamma_0^2 - \Lambda^{-2} \qquad \frac{d\Lambda}{dt} \geqslant 1 - \gamma_0^2\Lambda^2 \tag{6.13}$$

where the right-hand side is positive, by the inequality (A1.12) of appendix 1.

The commutator γ_0 is of order $\hbar$, so until the indeterminacy product becomes close to the minimum, the localization increases at a rate that is greater than 1. As the minimum is approached, an exact inequality is needed. Integration gives

$$d\Lambda/dt \geqslant \frac{(y-1)}{(y+1)|\gamma_0|} \tag{6.14}$$

where

$$y = \frac{1 + |\gamma_0|\Lambda_0}{1 - |\gamma_0|\Lambda_0}e^{2|\gamma_0|(t-t_0)} \tag{6.15}$$

where $\Lambda = \Lambda_0$ when $t = t_0$ and where $1 - |\gamma_0|\Lambda \geqslant 0$ by inequality (A1.12) of appendix 1. The states of the ensemble exponentially approach the special minimum indeterminacy wave packet of (A1.11) with equal QMS deviation for G_R and G_I as $t \to \infty$.

For the annihilation *and* creation operators, $|\gamma_0| = 1$ gives a lower bound on the rate of selflocalization, together with the asymptotic coherent state wave packet, which is the same for each.

7. Quasiclassical theory: phase space localization

From the general and linear theories of the earlier sections, it follows that when the effect of the Hamiltonian is negligible by comparison with the effect of the environment operators, the rate of localization in an ensemble is determined by commutation terms, of order $\hbar$ or smaller with either sign, and by correlation terms which always increase the localization. For the quasiclassical case the latter dominate, and for wide-open systems with negligible Hamiltonians and a sufficient number of arbitrary environment operators, the states localize to wave packets with Heisenberg indeterminacy products close to the minimum, which look like points in phase space on a classical scale.

Let G and L_j be selfadjoint operators for a wide-open system of m freedoms, in which the corresponding dynamical variables g and ℓ_j are smooth over 'classical' regions of phase space of dimension much larger than $\hbar^m$. Then the localization of a state procedes in three stages.

In the lowest order quasiclassical theory, the state vector is represented by a density distribution $D(x, y)$ in phase space, where

$$x = (x_1, \ldots x_m) \qquad y = (y_1, \ldots y_m) \tag{7.1}$$

are a complete set of coordinates and conjugate momenta and the distribution may or may not be singular. The density $D(x, y)$ for an individual state must be distinguished from the density obtained by averaging over the ensemble. Each pure state density corresponds to the quantum state of a single system of the ensemble and diffuses in the space of densities according to a diffusion equation obtained from the diffusion equation for state vectors.

The three stages of localization correspond to three levels of dispersion or localization of the density $D(x, y)$. In the first stage the density is so much dispersed that it is not confined to a region in which the dynamical variables can be approximated by dynamically linear variables, that is, by linear combinations of the canonical coordinates and momenta. In the second stage it is so confined, but the region is large compared to $\hbar^m$ so the dynamically linear theory applies in its classical version. In the third stage, the localization has effectively confined the system to a region of phase space comparable to $\hbar^m$, and the quantum version of the dynamically linear theory is required.

In the quasiclassical theory the quantum expectation of an operator G is approximated by the classical expectation over the density $D(x, y)$ of the corresponding classical dyanamical variable g,

$$\langle G \rangle = \langle \psi | G | \psi \rangle \approx \int d^m x \, d^m y D(x, y) g(x, y) = \langle g \rangle \tag{7.2}$$

where the last equality is a convenient notation for the classical expectation of a dynamical variable. Lower-case letters like g are used for dynamical variables and $\langle g \rangle$ is used for the *classical* expectation. This expectation of a dynamical variable over a density has to be distinguished from the mean M over the ensemble, just as the quantum expectation for a state has to be distinguished from the ensemble mean. In the quasiclassical theory a quantum product is approximated by the product of the corresponding classical dynamical variables and a commutator is aapproximated using the Poisson bracket:

$$AB \approx ab = a(x, y)b(x, y) \qquad [A, B] \approx i\hbar\{a, b\}. \tag{7.3}$$

Using these approximations, the equations (3.1) for the diffusion of $\langle g \rangle$ and (3.2) for the mean rate of change of $\sigma^2(g)$ become

$$d\langle g \rangle = -\tfrac{1}{2}\hbar^2 \langle \{\{g, \ell\}, \ell\} \rangle dt + \tfrac{1}{2} \mathrm{Re}(\sigma(g, \ell) \, d\xi) \qquad \text{(quasiclassical)} \tag{7.4}$$

and

$$\mathrm{M} \, d\sigma^2(g)/dt = -\tfrac{1}{2}\hbar^2 \langle \{\{g_\Delta^2, \ell\}, \ell\} \rangle - 2|\sigma(g, \ell)|^2 \qquad \text{(quasiclassical)} \tag{7.5}$$

where $\sigma(g, \ell) = \langle g_\Delta \ell \rangle$.

Now suppose the phase space distribution $D(x, y)$ extends over a region large compared to $\hbar^m$. The commutation terms are then usually negligible compared with the correlation terms. The localization therefore proceeds until the distribution has an area comparable to $\hbar$ for each pair of conjugate canonical variables in which the reduction operates, and proceeds indefinitely if the reduction takes place for a variable and not for its conjugate momentum.

Thus if the number of linearly independent environment operators is sufficient, and it is sufficient for almost all systems of classical size, the state vector localizes in phase space to a region of quantum dimensions, which looks like a point to classical eyes.

The precise form of the wave packet or coherent state to which the system localizes depends on the details of the environment operators, but a good approximation is obtained by expanding all the operators in a power series about the mean location in phase space, and stopping at the linear term.

The distribution is then given by the dynamically linear theory of section 5.

8. Quasiclassical and classical theory for WKB wavefunctions

This section relates the quasiclassical state diffusion theory to the elementary conventional quasiclassical theory, so as to give a geometrical phase space picture of the localization.

In the simplest WKB form of the quasiclassical theory of pure states of a closed or isolated quantum system of one freedom, a state vector is represented by a wavefunction in x representation of the form

$$\psi(x) = D(x)^{1/2} e^{iS(x)/\hbar} \tag{8.1}$$

where $D(x)$ is a classical density and $S(x)$ is a classical action. The phase space density is then confined to the region of a curve or wave front

$$(x, y) = (x, \partial S(x)/\partial x) \tag{8.2}$$

in the phase space, where y is conjugate to x. The phase space curve can be represented by a delta-distribution for $D(x, y)$, but it is more convenient and common to use the representation in terms of the phase curve defined by $S(x)$ or $y(x)$, with $D(x)$ to represent the density on the curve.

More generally, for a system of m freedoms, the density is confined to the region of an m-dimensional surface in the $2m$-dimensional phase space, given by the same equations with a vector interpretation. For example, the invariant tori of EBK quantization have this form, and wave packets which are intially confined to a region of phase space of size $\hbar^m$ are dispersed in time to this extended form by most Hamiltonians, for example, the kinetic energy, or by scattering. Thus for closed systems the natural classical analogues of quantum states are surfaces which satisfy the time-independent or time-dependent Hamilton–Jacobi equation.

Now we consider what happens when an isolated system with a WKB wavefunction is suddenly exposed to so strong an interaction with the environment that the effect of the Hamiltonian can thereafter be neglected and the system becomes wide open. Suppose there is one freedom and a single selfadjoint diffusion operator L, and let k be conjugate to ℓ.

Using the state diffusion equation (2.14) and the quasiclassical approximations, the change in the probability density is given by

$$
\begin{aligned}
dD(x) &= 2\operatorname{Re}(\psi^* \, d\psi) + d\psi^* \, d\psi \\
&= 2\operatorname{Re}\left(-\tfrac{1}{2}(\ell(x) - \langle\ell\rangle)^2 \psi^* \psi dt + (\ell(x) - \langle\ell\rangle)\psi^* \psi \, d\xi\right) + (\ell(x) - \langle\ell\rangle)^2 \psi^* \psi dt \\
&= 2(\ell(x) - \langle\ell\rangle)D(x) \, d\xi_{\mathrm{R}} \tag{8.3}
\end{aligned}
$$

where suffixes R and I represent real and imaginary parts.

To obtain the change in the action, use equations (8.1), (2.14) and (2.15) to give

$$dS(x)/\hbar = \text{Im}(d \ln \psi) = \text{Im}\left(d\psi/\psi - \tfrac{1}{2}(d\psi/\psi)^2\right)$$
$$= 0 + \text{Im}\left((\ell - \langle\ell\rangle)\,d\xi\right) + 0 \tag{8.4}$$

so that

$$dS(x) = \hbar(\ell - \langle\ell\rangle)\,d\xi_{\text{I}}. \tag{8.5}$$

The change in the action function produces a corresponding change in the classical limiting value of the dynamical variable k conjugate to ℓ given by

$$dk = d(\partial S/\partial\ell) = \hbar\,d\xi_{\text{I}}. \tag{8.6}$$

This diffusion is of order $\hbar$ times the diffusion in the x-density $D(x)$, and is independent of it. In (k, ℓ) representation, the shape and orientation of the phase space curve remain constant, but the whole curve diffuses slowly in the k-direction.

This result generalizes directly to vector dynamical variables ℓ with n components by interpreting ℓ and k as n-vectors, with n different Wiener processes $d\xi_r$ corresponding to the components ℓ_r.

A more general type of quasiclassical representation is as a sum over sheets s given by

$$\psi(x) = \sum_s \psi_s(x) = \sum_s D_s(x)^{1/2}\exp(iS_s(x)/\hbar) \tag{8.7}$$

where the functions

$$y_s(x) = \partial S_s(x)/\partial x \tag{8.8}$$

together describe a continuous Lagrangian m-manifold in the phase space, joined together at the classical turning points, which correspond to quantum caustics. In the classical limit the distribution is confined to this curve, with an x density defined by the $D_s(x)$. The mean of a dynamical variable g is

$$\langle g\rangle = \sum_s \int d^m x.D_s(x)g_s(x) \tag{8.9}$$

where $g_s(x)$ is the value of g on the sheet s at x. The equation for the change in the density $D_s(x)$ is then

$$dD_s(x) = 2(\ell_s(x) - \langle\ell\rangle)D_s(x)\,d\xi_R \tag{8.10}$$

giving the coupling between the sheets from the value of $\langle\ell\rangle$, which depends on the mean over all sheets.

Let

$$D_s = \int d^m x\, D_s(x) \qquad \langle g\rangle_s = D_s^{-1}\int d^m x\, D_s(x)g_s(x) \tag{8.11}$$

be the total density or expectation for sheet s and the expectation of g for sheet s. Then from (8.10)

$$dD_s = \int d^m x\, dD_s(x) = 2D_s(\langle\ell\rangle_s - \langle\ell\rangle)\,d\xi_R \tag{8.12}$$

so the coupling between the sheets changes the relative overall density of the sheets. The sheet expectations of those sheets whose expectation of ℓ is close to the overall expectation of ℓ, fluctuate little, but when the difference of expectations of ℓ is large the sheet expectations fluctuate a lot, and those particular sheets tend to lose density in the ensemble.

As shown in earlier sections, the asymptotic result of the state diffusion is localization to a surface of constant ℓ or to the quantum neighbourhood of a point in phase space.

9. Discussion

Previous papers [1–3, 7–10] built up a picture in which interaction with the environment produced localization, sometimes to an eigenstate, but more commonly to a state which is localized in phase space. Here that picture is confirmed and extended. A general theory is presented, lower bounds are put to rates of selflocalization, and bounds are put on asymptotic states.

For wide-open systems, localization to regions of phase space comparable in volume to $\hbar^m$ is the norm, so that for nearly wide-open systems, the limiting behaviour on a classical scale is the natural one of motion of points in phase space, satisfying Hamilton's equations, rather than WKB surfaces satisfying the Hamilton–Jacobi equation. For general open systems there is a physical conflict between the dispersion due to the Hamiltonian and the localization due to the state diffusion.

Perhaps the most interesting result is that there is a purely classical limit of state diffusion in which the state of a single system is represented by a density in phase space, which diffuses according to the equation (8.10), in which Planck's constant does not appear. The diffusion depends only on the dynamical variables corresponding to the environment operators, and their means. Because of the appearance of the mean there are strong non-local effects, including localization towards distributions of quantum dimensions that appear to classical eyes like phase space points. Localization can be a classical effect. For general open systems there is simultaneous Liouville and diffusion evolution of the phase space density given by

$$dD(x, y) = \{h, D\}dt + 2\sum_{j}(\ell_j(x, y) - \langle\ell_j\rangle)D(x, y))\,d\xi_{jR} \qquad (9.1)$$

where $h(x, y)$ is the classical Hamiltonian, $D(x, y)$ is the classical phase space density, which may be singular, $\{.,.\}$ is the Poisson bracket, and $d\xi_{jR}$ are independent real Wiener processes with

$$(d\xi_{jR})^2 = dt/2 \qquad M\,d\xi_{jR} = 0. \qquad (9.2)$$

Although this theory is entirely classical, there is little possibility of seeing the effects for macroscopic bodies in our era, because there are so many interactions with the environment that the localization to a phase point is too fast. It would have happened long ago, giving today the ordinary classical picture of phase points moving according to Hamilton's equations. It is conceivable that classical localization of large bodies may have been significant in the very early Universe. However, for some of today's microscopic systems there may well be situations in which the localization can be treated classically, provided there is no need for detail in phase space on a scale of Planck's constant.

Appendix 1. Bounds on QMS deviations

Let B and C be any two selfadjoint operators, with constant commutator

$$[B, C] = i\gamma \neq 0 \tag{A1.1}$$

and let $c = \gamma^2/4$. By the Heisenberg inequality,

$$\sigma^2(B)\sigma^2(C) = (\Delta B)^2(\Delta C)^2 \geq c. \tag{A1.2}$$

So for any positive integer n, the sum of the nth powers of the QMS deviations satisfies the inequality

$$\left(\sigma^2(B)\right)^n + \left(\sigma^2(C)\right)^n \geq \left(\sigma^2(B)\right)^n + c^n / \left(\sigma^2(B)\right)^n. \tag{A1.3}$$

The minimum value of the right-hand side is attained when

$$\left(\sigma^2(B)\right)^n = c^{n/2} \tag{A1.4}$$

so that

$$\sigma^2(B) = \sigma^2(C) = |\gamma|/2. \tag{A1.5}$$

This minimum is therefore only achieved by the specific minimum indeterminacy wave packet for which the QMS deviations of B and C are equal. In general, for arbitarary states, the sum on the left of (A1.3) therefore satisfies the inequality

$$\left(\sigma^2(B)\right)^n + \left(\sigma^2(C)\right)^n \geq 2c^{n/2} = 2(|\gamma|/2)^n = 2(|[B, C]|/2)^n. \tag{A1.6}$$

The quantum correlations between selfadjoint operators also satisfy the inequalities. From the commutation relations

$$\sigma(B, C) - \sigma(B, C)^* = \sigma(B, C) - \sigma(C, B) = i\gamma \tag{A1.7}$$

so

$$|\sigma(B, C)| = |\sigma(C, B)| \geq \gamma/2. \tag{A1.8}$$

If $B = G_R$ and $C = G_I$ are the real and imaginary parts of a non-selfadjoint operator G, then

$$[G_R, G_I] = i\gamma \qquad [G^\dagger, G] = -2\gamma \tag{A1.9}$$

and from equation (2.8),

$$\sigma^2(G) + \gamma = \sigma^2(G_R) + \sigma^2(G_I) \tag{A1.10}$$

is minimized when

$$\sigma^2(G_R) = \sigma^2(G_I) = |\gamma|/2 \tag{A1.11}$$

and so by the definition (2.12) for the ensemble localization Λ of a non-selfadjoint operator,

$$\Lambda \leq \frac{1}{|\gamma|} = \frac{2}{|[G^\dagger, G]|} \tag{A1.12}$$

an overall limit on the localization of a non-selfadjoint operator.

Appendix 2. Equivalence and linear dependence of environment operators

Equivalent sets of environment operators are those that result in the same quantum state difffusion. Here I show directly that in a finite-dimensional state space, linearly dependent finite sets of environment operators are equivalent to a smaller number of linearly independent environment operators. As pointed out in [1], a set

$$L_j \qquad (j = 1, \ldots n) \tag{A2.1}$$

of environment operators for quantum state diffusion is equivalent to any other set

$$L'_k \qquad (k = 1, \ldots n) \tag{A2.2}$$

related to the first by a unitary transformation

$$L_j = \sum_k L'_k u_{kj} \qquad (j, k = 1, \ldots n). \tag{A2.3}$$

The scalar product of two operators is defined in the usual way as

$$(A, B) = \text{Tr}(A^\dagger B). \tag{A2.4}$$

The scalar product then defines Hermitean scalar product matrices

$$s_{jj'} = (L_j, L_{j'}) \qquad s'_{kk'} = (L'_k, L'_{k'}). \tag{A2.5}$$

related by the unitary transformation

$$s_{jj'} = \sum_{k,k'} u^*_{jk} s'_{kk'} u_{k'j'} \tag{A2.6}$$

where all suffices range from 1 to n. The unitary transformation can be chosen to diagonalize the Hermitean scalar product matrix $s'_{kk'}$, so that all the cross products are zero.

Now suppose that the number of linearly independent L_j is $n_0 < n$. Then this will also be true of the L'_k, and the only way this can happen is for $n - n_0$ of the L'_k to have zero norm, so they are identically zero and have no effect on the diffusion. Thus every state diffusion equation in a finite-dimensional state space is equivalent to a state diffusion equation with linearly independent environment operators L_j.

In a practical problem with an infinite-dimensional phase space, a finite-dimensional subspace is usually an adequate approximation. For example, free particles can be put in a large box, with a momentum cutoff.

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